

Homework 7

Math 121A, Fall 2025.

Due: Wednesday, Oct 15, 11:59 PM

Unless otherwise stated, all solutions require clear, well-written justification. You must explicitly credit any collaborators for each problem.

Problem 1. (*Return of the Laplacian*) We call a twice differentiable function $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ *harmonic* if it solves Laplace's equation:

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

We often call the operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ the *Laplacian*, and it appears in electromagnetism, gravity, quantum mechanics, thermodynamics...

Show that if $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic on a domain Ω ,¹ then u and v are harmonic.

Problem 2. Find the real $u(x,y)$ and imaginary $v(x,y)$ parts of the following complex functions. Decide if each function is holomorphic.

- (a) $|z|$.
- (b) $\sin z$.
- (c) $\bar{z}$.
- (d) $\frac{1}{z}$.

Problem 3. Problem 14.2.28

Problem 4. (a) Problem 14.3.3

(b) Problem 14.3.6

(c) Problem 14.3.12

Problem 5. Use Cauchy's integral formula to evaluate the following integrals for the given contour.

(a) Let C be the circle $|z - 3| = 2$, oriented positively:

$$\int_C \frac{e^z + \sin z}{z} dz.$$

(b) Let $\tilde{C}$ be the circle $|z - 2| = 3$, oriented positively:

$$\int_{\tilde{C}} \frac{e^z + \sin z}{z} dz.$$

¹A deep fact in complex analysis is that holomorphic functions are infinitely differentiable, so we may take as many derivatives as we please.

(c) Let C' be the unit circle $|z| = 1$, oriented negatively:

$$\int_{C'} \frac{z^2 e^z}{ez + i} dz.$$

Problem 6. (*Fundamental theorem of calculus*)² If $f(z)$ is holomorphic on a region Ω and $f(z) = \frac{dF}{dz}(z)$ on R , then F is called an *antiderivative* of f . Show that if f has an antiderivative on Ω , then for any contour γ in Ω beginning at z_1 and ending at z_2 ,

$$\int_{\gamma} f(z) dz = F(z_2) - F(z_1).$$

(hint: use the chain rule and the fundamental theorem of calculus for real functions).

Use this to evaluate the integral

$$\int_{\gamma} e^z + \sin z \, dz,$$

where $\gamma(t) = e^{it}$, $t \in [0, \pi]$ is a half-circle of radius 1.

Problem 7. (*Primer for the residue theorem*)

(a) Let γ be any circle centered at the origin with positive (counterclockwise) orientation. Evaluate the integrals

$$\int_{\gamma} z^n dz$$

for all integers n .

(b) Using part (a), show that if f is of the form

$$f(z) = g(z) + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + \frac{a_n}{z^n}$$

where $g(z)$ is holomorphic on and inside the circle γ , then

$$\oint_{\gamma} f(z) dz = 2\pi i a_1.$$

²This is essentially the same as the fundamental theorem of line integrals for conservative vector fields.